Fixed point theorems for $\psi-$contractions on a G- metric space and consequences

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Abstract :-

We prove some fixed point theorems for self mappings satisfying some kind of contractive type conditions on complete G-metric spaces and obtain results of Shatanawi[15], Sushanta Kumar Mohanta[16] and Vats et al.[17] as corollaries.

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1. Introduction:-

The study of metric fixed point theory plays an important role because the study finds applications in many important areas as diverse as differential equations, operation research, mathematical economics and the like. Different generalizations of the usual notion of a metric space were proposed by several mathematicians such as Gahler [3,4] (called 2-metric spaces) and Dhage [1,2] (called D-metric spaces). K.S.Ha et al. [5] have pointed out that the results cited by Gahler are independent, rather than generalizations, of the corresponding results in metric spaces. Moreover, it was shown that Dhage's notion of D-metric space is flawed by errors and most of the results established by him and others are invalid. These facts determined Mustafa and Sims [11] to introduce a new concept in the area, called G-metric space. Recently, Mustafa et al. studied many fixed point theorems for mappings satisfying various contractive conditions on complete G-metric spaces; see [8-13]. Subsequently, some authors like Renu Chugh et al.[14], W.Shatanawi [15] have generalized some results of Mustafa et al. [7-8] and studied some fixed point results for self-mappings in a complete G-metric space under some contractive conditions related to a non-decreasing $\phi:[0,+\infty)\to[0,+\infty)$ with $\lim_{n\to\infty}\phi^n(t)=0$ for all $t\in(0,+\infty)$.

Kumara Swamy and Phaneendra[6] and Sushanta Kumar Mohanta[16] proved some fixed point theorems for self-mappings on complete G-metric spaces.

In this paper we introduce ψ -contractions in G-metric spaces, prove fixed point results for such maps and obtain results of Shatanawi [15], Sushanta Kumar Mohanta [16] and Vats et al.[17] as corollaries.

2 Preliminaries:-

We begin by briefly recalling some basic definitions and results for G-metric spaces that will be needed in the sequel.

Definition 2.1 :-(Mustafa and Sims [7]) Let X be a non–empty set, and let $G: X\times X\times X\to R^+$ be a function satisfying the following axioms:

$(G1)$ $G(x,y,z)=0$ if $x=y=z$,
$(G2)$ $0< G(x,x,y)$, for all $x,y\in X$, with $x\neq y$,
$(G3)$ $G(x,x,y)\leq G(x,y,z)$,
for all $x,y,z\in X$, with $z\neq y$,
$(G4)$ $G(x,y,z)=G(\pi(x,z,y))$
where π is a permutation in $\{x,y,z\}$,
$(G5)$ $G(x,y,z)\leq G(x,a,a)+G(a,y,z)$,
for all $x,y,z,a\in X$, (rectangle inequality).

Then the function G is called a generalized metric, or, more specifically a G- metric on X , and the pair (X,G) is called a G-metric space.

Example 2.2:- (Mustafa and Sims [7]) Let R be the set of all real numbers. Define. $G: R\times R\times R\to R^+$ by

$$G(x,y,z)=|x-y|+|y-z|+|z-x|,\ \text{for all } x,y,z\in X.$$

Then it is clear that (R,G) is a G-metric space.

We use the following proposition in the Sequel without explicit mention.

Proposition 2.3:- (Mustafa and Sims [7]) Let (X,G) be a G-metric space. Then for any $x,y,z,$ and $a\in X$, it follows that

(1) if $G(x,y,z)=0$ then $x=y=z$,
(2) $G(x,y,z)\leq G(x,x,y)+G(x,x,z)$,
(3) $G(x,y,y)\leq 2G(y,x,x)$, $\longrightarrow$ $(2.3.1)$
(4) $G(x,y,z)\leq G(x,a,z)+G(a,y,z)$,
(5) $G(x,y,z)\leq \frac{2}{3}(G(x,y,a)+G(x,a,z)+G(a,y,z))$,
(6) $G(x,y,z)\leq G(x,a,a)+G(y,a,a)+G(z,a,a)$.

Definition 2.4: (Mustafa and Sims [7]) Let (X,G) be a G-metric space, let $\{x_n\}$ be a sequence of points of X , we say that $\{x_n\}$ is G-convergent to x if $\lim_{n,m\to\infty}G(x,x_n,x_m)=0$; that is, for any $\in>0$, there exists $n_0\in N$ such that $G(x,x_n,x_m)<\in$, for all $n,m,\geq n_0$. We refer to x as the limit of the sequence $\{x_n\}$ and write $x_n\longrightarrow x$.

The following proposition is used in the section 3.

Proposition 2.5:- (Mustafa and Sims [7]) Let (X,G) be a G-metric space. Then, the following are equivalent:

(1) $\{x_n\}$ is G-convergent to x.
(2) $G(x_n,x_n,x)\to 0,$ as $n\to\infty$.
(3) $G(x_n,x,x)\to 0,$ as $n\to\infty$.
(4) $G(x_m,x_n,x)\to 0,$ as $m,n\to\infty$.

Definition 2.6:- (Mustafa and Sims [7]) Let (X,G) be a G-metric space, sequence $\{x_n\}$ is called G-Cauchy if given $\in>0,$ there is $n_0\in N$ such that $G(x_n,x_m,x_l)<\in,$ for all $n,m,l\geq n_0$ that is if $G(x_n,x_m,x_l)\to 0$ as $n,m,l\to\infty.$

Proposition 2.7:- (Mustafa and Sims [7]) In a G-metric space (X,G) , the following are equivalent:

(1) The sequence $\{x_n\}$ is G-Cauchy.
(2) For every $\in>0,$ there exists $n_0\in N$ such that $G(x_n,x_m,x_m)<\in$ for all $n,m\geq n_0.$

Definition 2.8:- (Mustafa and Sims [7]) Let (X,G) and (X',G') be G-metric spaces and let $f:(X,G)\to(X',G')$ be a function, then *f* is said to be G-continuous at a point $a\in X$ if given $\in>0,$ there exists $\delta>0$ such that $x,y\in X;G(a,x,y)<\delta$ implies $G'(f(a),f(x),f(y))<\in.$ A function f is G-continuous on X if and only if it is G-continuous at all $a\in X.$

Proposition 2.9:- (Mustafa and Sims [7]) Let (X,G) and (X',G') be G-metric spaces, then a function $f:X\to X'$ is G-continuous at a point $x\in X$ if and only if it is G-sequentially continuous at x ; that is, whenever $\{x_n\}$ is G-convergent to $x,\{f(x_n)\}$ is G-convergent to $f(x)$.

Proposition 2.10:- (Mustafa and Sims [7]) Let (X,G) be a G-metric space. Then, the function $G(x,y,z)$ is continuous in all variables.

Definition 2.11:- (Mustafa and Sims [7]) A G-metric space (X,G) is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in (X,G) is G-convergent in (X,G).

3 Main results:-

We first prove a lemma followed by a notation and a definition.

3.1.Lemma:- Suppose $\psi:[0,\infty)\to[0,\infty)$ is such that ψ is continuous, increasing and $\sum\psi^n(t)<\infty$ for $t>0.$

Then (i) $\psi(t)=0$ if $t=0$ and (ii) $\psi(t)<t$ for $t>0.$

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Proof:-(i) Suppose $\psi(0)=0,\ k>0$

$$0<k \Rightarrow \psi(0)\le\psi(k)\Rightarrow$$
$$k\le\psi(k)\Rightarrow\psi(k)\le\psi^2(k)\Rightarrow$$
$$k\le\psi(k)\le\psi^2(k),$$

hence by induction

$$k\le\psi(k)\le\psi^2(k).......\le\psi^n(k).....$$

$\therefore \sum\psi^n(k)$ is ∞,

a contradiction

$\therefore \psi(0)=0.$

(ii) $t\le\psi(t)$ for some $t>0\Rightarrow$

$$\psi(t)\le\psi^2(t)\Rightarrow t\le\psi(t)\le\psi^2(t)\Rightarrow$$
$$t\le\psi(t)\le\psi^2(t)\le\psi^3(t)$$
$$\Rightarrow t\le\psi(t)\le\psi^2(t).......\le\psi^n(t)\le.....$$

$\Rightarrow \sum\psi^n(t)$ is ∞,

a contradiction

$\therefore \psi(t)<t$ for $t>0$.

Notation:- $\Psi=\{\psi:[0,\infty)\to[0,\infty)\}$ such that

ψ is continuous, increasing and

$\sum\psi^n(t)<\infty$ for $t>0$.

3.2. Definition:- Let (X,G) be a complete G-metric space , $\psi\in\Psi$, and $T:X\to X$. Suppose $F(x,y,z)$ is a function of $G(x,y,z),G(Tx,Ty,Tz)$ and their variants. Assume that

$$G(Tx,Ty,Tz)\le\psi(F(x,y,z))\ \forall x,y,z\in X.$$

Then T is called a $\psi-$contraction.

3.3.Theorem:- Let (X,G) be a complete G-metric space and $\psi\in\Psi$. Suppose that $T:X\to X$ satisfies.

$$(3.3.1)\ G(Tx,Ty,Tz)\le\psi\big(\max\{G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz)\}\big)\ \forall x,y,z\in X.$$

Then T has a unique fixed point .

Proof:- Let $x_0\in X$, write $x_{n+1}=Tx_n, n=0,1,2,....$

We first show that $\{x_n\}$ is a Cauchy sequence.

Put $x=x_n,\ y=z=x_{n+1}$ in $(3.3.1)$. Then

$$G(x_{n+1},x_{n+2},x_{n+2})\le\psi\big(\max\{G(x_n,x_{n+1},x_{n+1}),G(x_n,x_{n+1},x_{n+1}),G(x_{n+1},x_{n+2},x_{n+2}),G(x_{n+1},x_{n+2},x_{n+2})\}\big)$$
$$=\psi\big(\max\{G(x_n,x_{n+1},x_{n+1}),G(x_{n+1},x_{n+2},x_{n+2})\}\big)$$
$$\therefore \alpha_{n+1}\le\psi(\max\{\alpha_n,\alpha_{n+1}\}),$$

where $\alpha_n=G(x_n,x_{n+1},x_{n+1})$

Now $\alpha_n\le\alpha_{n+1}\Rightarrow\alpha_{n+1}\le\psi(\alpha_{n+1})<\alpha_{n+1}$

a contradiction.

$\therefore \alpha_{n+1}\le\alpha_n$

$\therefore \{\alpha_n\}$ is a decreasing sequence.

$\therefore$ Suppose $(\alpha_n)\downarrow r$.

Then $\psi(r)\downarrow\psi(r)$ (since ψ is continuous)

$\therefore r\le\alpha_{n+1}\le\psi(\alpha_n)<\alpha_n$.

As $n\to\infty$, we get $r\le\psi(r)\le r$.

$\therefore\psi(r)=r$ so that $r=0$.

Now $\alpha_{n+1}\le\psi(\alpha_n)$

$\le\psi^2(\alpha_{n-1})\le\ldots\le\psi^{n+1}(\alpha_0)\longrightarrow(3.3.2)$

Suppose $n<m$.

Then by (G_5)

we get

$$\begin{aligned}G(x_n,x_m,x_m)&\le G(x_n,x_{n+1},x_{n+1})\\&+G(x_{n+1},x_{n+2},x_{n+2})+\ldots+\\&G(x_{m-1},x_m,x_m)=\alpha_n+\alpha_{n+1}+\ldots+\alpha_{m-1}\\&\le\psi^n(\alpha_0)+\psi^{n+1}(\alpha_0)+\ldots\\&\ldots\ldots\ldots+\psi^{m-1}(\alpha_0)\ \ (\text{from }(3.3.2))\\&\to 0\text{ as }m,n\to\infty\ \left(\text{since }\sum\psi^n(t)<\infty\right)\end{aligned}$$

$\therefore G(x_n,x_m,x_m)\to 0$

as $m,n\to\infty$.

$\therefore\{x_n\}$ is a Cauchy sequence.

Suppose $\{x_n\}\to p$.

Now

$$\begin{aligned}G(x_{n+1},Tp,Tp)&=G(Tx_n,Tp,Tp)\\&\le\psi\left(\max\{G(x_n,p,p),G(x_n,x_{n+1},x_{n+1}),\right.\\&\left.G(p,Tp,Tp),G(p,Tp,Tp)\}\right)\end{aligned}$$

On letting $n\to\infty$, we get

$$\begin{aligned}G(p,Tp,Tp)&\le\psi\left(\max\{G(p,p,p),G(p,p,p),\right.\\&\left.G(p,Tp,Tp),G(p,Tp,Tp)\}\right).\\&=\psi(G(p,Tp,Tp))\end{aligned}$$

$\therefore G(p,Tp,Tp)\le\psi(G(p,Tp,Tp))$

$\therefore G(p,Tp,Tp)=0$

$\therefore Tp=p$.

$\therefore p$ is a fixed point of T.

Now we obtain Banach contraction principle for $\psi-$contractions on a G-metric space as a corollary from Theorem 3.3

3.4.Theorem :- (Banach contraction principle for $\psi-$contractions) Let (X,G) be a complete G-metric space and T be a self map on *X.* Suppose there exists $\psi\in\Psi$, such that

$$(3.4.1)\leftarrow G(Tx,Ty,Tz)\le\psi(G(x,y,z))\forall x,y,z\in X.$$

Let

$x_0\in X$ and write $x_n=T^n(x_0)$

$=T(T^{n-1}(x_0))$ for $n=1,2,\ldots$

Then the sequence $\{x_n\}$ is a Cauchy sequence in *X*.

If further *X* is complete then $\{x_n\}$ converges to say, x and x is a fixed point of *T*. Also *T* has a unique fixed point.

Proof:- Clearly $(3.4.1)\Rightarrow(3.3.1)$. Hence the result follows from Theorem 3.3

Now we obtain the following corollary from Theorem 3.4

3.5. corollary :-(Vats et al.[17], lemma 1 and Shatanawi[15], corollary 3.4) Let (X,G) be a complete G-metric space and *0<q<1.* Suppose *T* is a self map on *X* satisfying

$$(3.5.1)\ \ G(Tx,Ty,Tz)\le q\ G(x,y,z)\forall x,y,z\in X.$$

Let $x_0 \in X$ and write

$x_{n+1} = Tx_n, n = 0,1,2,...$ in X.

Then $\{x_n\}$ Cauchy sequence.

If $x_n \to x$ then x is the unique fixed point of T.

Proof:- Take $\psi(t) = qt$ for $t \geq 0$. Now the result follows from Theorem 3.4

Now we state and prove a simple corollary as Theorem 3.3

3.6.Corollary :- Suppose (X, G) is a complete G-metric space and *0<k<1.* Assume that $T : X \to X$ satisfies

$$G(Tx,Ty,Tz) \leq k \max\{G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz)\} \quad \forall x,y,z \in X.$$

Then T has a unique fixed point.

Proof:- Write

$\psi(t) = kt$ for $t > 0$. Then $\psi \in \Psi$.

Now the result follows from Theorem 3.3

The following theorem can be proved on the lines of proof of theorem 3.3

3.7.Theorem:- Let (X, G) be a complete G-metric space and $\psi \in \Psi$. Suppose $T : X \to X$ satisfies

$$(3.7.1) \quad G(Tx,Ty,Tz) \leq \psi\left(\max\{G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), G(Tx,y,z)\}\right) \quad \forall x,y,z \in X.$$

Then T has a unique fixed point .

An immediate corollary of theorem (3.7) is as f

3.8.Corollary:- (Shatanawi [15], corollary 3.7). Let (X, G) be a complete G-metric space and $0 < k < 1$. Suppose that $T : X \to X$ satisfies.

$$(3.8.1)\, G(Tx,Ty,Tz) \leq k\left(\max\{G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), G(Tx,y,z)\}\right. \quad \forall x,y,z \in X.$$

Then T has a unique fixed point .

Proof:- Write

$\psi(t) = kt$ for $t > 0$. Then $\psi \in \Psi$ and (3.8.1) $\Rightarrow$ (3.7.1)

Hence the result follows from Theorem 3.7

3.9.Theorem:- Let (X, G) be a complete G-metric space and let $T : X \to X$ be such that

$$G(T(x),T(y),T(z))$$
$$\le \psi\Big(\max\{\max\{G(x,y,z),G(x,T(x),T(x)),G(y,T(y),T(y)),G(z,T(z),T(z))\},$$
$$\frac{1}{2}\max\{G(x,T(y),T(y)),G(y,T(x),T(x)),G(y,T(z),T(z)),G(z,T(y),T(y)),$$
$$G(z,T(x),T(x)),G(x,T(z),T(z))\}\}\Big)\longrightarrow(3.9.1)$$

for some $\psi\in\Psi$ and for all $x,y,z\in X$.

Then T has a unique fixed point in X.

Proof:-

Let $x_0\in X$ and $x_n=T^n(x_0)$

Take $x=x_n, y=z=x_{n+1}$ in $(3.9.1)$.

Since $x_{n+1}=T(x_n), x_{n+2}=T(x_{n+1})$,

we have

$$G(x_{n+1},x_{n+2},x_{n+2})=G(T(x_n),T(x_{n+1}),T(x_{n+1}))$$
$$\le\psi\Big(\max\{\max\{G(x_n,x_{n+1},x_{n+1}),G(x_n,T(x_n),T(x_n)),G(x_{n+1},T(x_{n+1}),T(x_{n+1})),G(x_{n+1},T(x_{n+1}),T(x_{n+1}))\},$$
$$\frac{1}{2}\max\{G(x_n,T(x_{n+1}),T(x_{n+1})),G(x_{n+1},T(x_n),T(x_n)),G(x_{n+1},T(x_{n+1}),T(x_{n+1})),G(x_{n+1},T(x_{n+1}),T(x_{n+1})),$$
$$G(x_{n+1},T(x_n),T(x_n)),G(x_n,T(x_{n+1}),T(x_{n+1}))\}\}\Big)$$

$$=\psi\Big(\max\{\max\{G(x_n,x_{n+1},x_{n+1}),G(x_n,x_{n+1},x_{n+1}),G(x_{n+1},x_{n+2},x_{n+2}),G(x_{n+1},x_{n+2},x_{n+2})\},$$
$$\frac{1}{2}\max\{G(x_n,x_{n+2},x_{n+2}),G(x_{n+1},x_{n+1},x_{n+1}),G(x_{n+1},x_{n+2},x_{n+2}),G(x_{n+1},x_{n+2},x_{n+2}),$$
$$G(x_{n+1},x_{n+1},x_{n+1}),G(x_n,x_{n+2},x_{n+2})\}\}\Big)$$

$$\le\psi\Big(\max\{\max\{G(x_n,x_{n+1},x_{n+1}),G(x_{n+1},x_{n+2},x_{n+2})\},$$
$$\frac{1}{2}\max\{G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},x_{n+2},x_{n+2}),G(x_{n+1},x_{n+2},x_{n+2})\}\}\Big)\qquad(\text{by } G_5)$$

$$\le\psi\Big(\max\{\max\{G(x_n,x_{n+1},x_{n+1}),G(x_{n+1},x_{n+2},x_{n+2})\}$$
$$\max\Big\{\frac{G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},x_{n+2},x_{n+2})}{2},\frac{1}{2}.G(x_{n+1},x_{n+2},x_{n+2})\Big\}\Big\}\Big)$$

$$=\psi\Big(\max\Big\{\max\{\alpha_n,\alpha_{n+1}\},\max\Big\{\frac{\alpha_n+\alpha_{n+1}}{2},\frac{\alpha_{n+1}}{2}\Big\}\Big\}\Big),\text{ where } \alpha_n=G(x_n,x_{n+1},x_{n+1})$$
$$\therefore \alpha_{n+1}\le\psi\Big(\max\Big\{\max\{\alpha_n,\alpha_{n+1}\},\frac{\alpha_n+\alpha_{n+1}}{2}\Big\}\Big).$$
$$=\psi(\max\{\alpha_n,\alpha_{n+1}\})$$

Now $\alpha_n<\alpha_{n+1}\Rightarrow\alpha_{n+1}\le\psi(\max\{\alpha_n,\alpha_{n+1}\})$

$=\psi(\alpha_{n+1})<\alpha_{n+1}$ a contradiction.

$\therefore \alpha_{n+1}\le\alpha_n\forall n$

$\therefore\{\alpha_n\}\downarrow r>0$ and hence $\psi(\alpha_n)\downarrow s(\text{say})$

$\therefore\alpha_{n+1}\le\psi(\alpha_n).\longrightarrow(3.9.2)$

$\therefore$ On letting $n\to\infty$, we get $r\le s\le\psi(\alpha_n)\to\psi(r)$

$\therefore r\le\psi(r)<r$ if $r>0$, a contradiction.

$\therefore r=0$ and hence $s=0$

Now we show that $\{\alpha_n\}$ is Cauchy.

From $(3.9.2)$, by induction, it follows that $\alpha_{n+1}\le\psi^{n+1}(\alpha_0)$.

Now by (G_5)

$G(x,y,z)\le G(x,a,a)+G(a,y,z)$

$\therefore G(x_n,x_{n+2},x_{n+2})\le G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},x_{n+2},x_{n+2})$

Suppose $n<m$. By introduction,

$$G(x_n,x_m,x_m)\le G(x_n,x_{n+1},x_{n+1})+G(x_{n+1},x_{n+2},x_{n+2})+.....+G(x_{m-1},x_m,x_m)$$
$$=\alpha_n+\alpha_{n+1}+\alpha_{n+2}+\alpha_{n+3}+.......+\alpha_{m-1}$$
$$\therefore G(x_n,x_m,x_m)\le\sum_{i=0}^{m-n-1}\alpha_{n+i}$$
$$\le\sum_{i=0}^{m-n-1}\psi^{(n+i)}(\alpha_0)\to 0 \text{ as } m,n\to\infty$$

Since $\sum\psi^n(t)<\infty$ for $t>0$, it follows that $\{x_n\}$ is Cauchy.

$\therefore\{x_n\}\to a$ limit p (say)

Consider

$$G(x_{n+1},T(p),T(p))=G(T(x_n),T(p),T(p))$$

$$\le\psi\Big(\max\{\max\{G(x_n,p,p),G(x_n,x_{n+1},x_{n+1}),G(p,T(p),T(p)),G(p,T(p),T(p))\},$$
$$\frac{1}{2}\max\{G(x_n,T(p),T(p)),G(p,x_{n+1},x_{n+1}),G(p,T(p),T(p)),G(p,T(p),T(p)),$$
$$G(p,x_{n+1},x_{n+1}),G(x_n,T(p),T(p))\}\}\Big)$$

On letting $n\to\infty$, we get

$$G(p,T(p),T(p))\le\psi\Big(\max\{G(p,T(p),T(p)),\frac{1}{2}\max\{G(p,T(p),T(p)),0,G(p,T(p),T(p))\}\}\Big)$$

$=\psi(G(p,T(p),T(p))$

$\therefore Tp=p$ so that p is a fixed point of T.

Now we show that T has unique fixed point.

Suppose p and q are fixed points of T

Put $x=p$ and $y=z=q$ in (3.9.1)

Then

$$G(T(p),T(q),T(q))\le\psi(\max\{\max\{G(p,q,q),G(p,T(p),T(p)),G(q,T(q),T(q)),G(q,T(q),T(q))\},$$

$$\frac{1}{2}\max\{G(p,T(q),T(q)),G(q,T(p),T(p)),G(q,T(q),T(q)),G(q,T(q),T(q)),G(q,T(p),T(p)),G(p,T(q),T(q))\}\})$$

$$G(p,q,q)\le\psi(\max\{\max\{G(p,q,q),G(p,p,p),G(q,q,q),G(q,q,q)\},$$

$$\frac{1}{2}\max\{G(p,q,q),G(q,p,p),G(q,q,q),G(q,q,q),G(q,p,p),G(p,q,q)\}\})$$

$$=\psi(\max\{G(p,q,q),\frac{1}{2}\max\{G(p,q,q),G(q,p,p)\}\})$$

$$=\psi\left(\max\left\{G(p,q,q),\frac{G(p,q,q)}{2},\frac{2G(p,q,q)}{2}\right\}\right)$$

$$=\psi(G(p,q,q))$$

$$\therefore G(p,q,q)\le\psi(G(p,q,q))$$

$$\therefore G(p,q,q)=0.$$

$$\therefore p=q.$$

Thus T has unique fixed point .

3.10.Theorem :-(Sushanta Kumar Mohanta [16] , Theorem 3.1 **)**

Let *(X,G)* be a complete G-metric space and $T:X\to X$ be such that

$$G(T(x),T(y),T(z)\le a\,G(x,y,z)+b\,G(x,T(x),T(x))+c\,G(y,T(y),T(y))+d\,G(z,T(z),T(z))$$
$$+e\max\{G(x,T(y),T(y)),G(y,T(x),T(x)),G(y,T(z),T(z)),G(z,T(y),T(y)),G(z,T(x),T(x)),G(x,T(z),T(z))\}\to(3.10.1)$$

for all $x,y,z\in X$, where a ,b,c,d,e ≥ 0 with $a+b+c+d+2e<1$.

Then T has a unique fixed point in X.

Proof:- Write

$\lambda=a+b+c+d+2e$ so that $\lambda<1$.

Define $\psi(t)=\lambda t$ for $t\ge 0$.

Then $\psi\in\Psi$.

Write

$$\alpha=\max\{G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz)\}$$

and $\beta=\max\{G(x,Ty,Ty),G(y,Tx,Tx),G(y,Tz,Tz),G(z,Ty,Ty),G(z,Tx,Tx),G(x,Tz,Tz)\}$

Then, from (3.10.1), we have

$$G(Tx,Ty,Tz)\le(a+b+c+d)\alpha+2e.\frac{1}{2}\beta$$
$$\le(a+b+c+d+2e)\max\left\{\alpha,\frac{1}{2}\beta\right\}$$
$$=\lambda\max\left\{\alpha,\frac{1}{2}\beta\right\}$$
$$=\psi\left(\max\left\{\alpha,\frac{1}{2}\beta\right\}\right)$$

so that (3.9.1) holds. Consequently, by Theorem 3.9,

T has unique fixed point.

The following fixed point Theorem for $\psi-$contractions on a complete G-metric space can be proved on the lines of proof of Theorem 3.3

3.11.Theorem:- Let *(X,G)* be a complete G-metric space and $\psi \in \Psi$. Suppose *T* $:X \to X$ satisfies the following:

$$(3.11.1)\ G(T(x),T(y),T(z)) \le \psi\Big(\max\{G(x,T(x),T(x)),G(x,T(y),T(y)),G(x,T(z),T(z)), G(y,T(y),T(y)),G(y,T(x),T(x)),G(y,T(z),T(z)), G(z,T(z),T(z)),G(z,T(x),T(x)),G(z,T(y),T(y))\}\Big)\ \forall x,y,z \in X.$$

If $\psi(t) < \frac{t}{2}$ for all $t > 0$, then *T* has a unique fixed point.

3.12.Corollary :- (VATS et al.[17], Theorem 3.1) Let *(X,G)* be a complete G-metric space. Suppose $0 < k < \frac{1}{2}$ and *T* $:X \to X$ satisfies the following:

$$(3.12.1)\ G(T(x),T(y),T(z)) \le k\max\{G(x,T(x),T(x)),G(x,T(y),T(y)),G(x,T(z),T(z)), G(y,T(y),T(y)),G(y,T(x),T(x)),G(y,T(z),T(z)), G(z,T(z),T(z)),G(z,T(x),T(x)),G(z,T(y),T(y))\}\ \forall x,y,z \in X.$$

Then *T* has a unique fixed point.

Proof:- Take $\psi(t) = kt$ for $t \ge 0$.

Then $(3.12.1) \Rightarrow (3.11.1)$ and hence the result follows from Theorem 3.11

The following theorem can be proved on the lines of proof of Theorem3.3

3.13 Theorem:- Let *(X,G)* be a complete G-metric space and $\psi \in \Psi$. Suppose *T* $:X \to X$ satisfies the following:

$$(3.13.1)\ G(T(x),T(y),T(z)) \le \psi\Big(\max\{G(x,T(x),T(x))+G(y,T(y),T(y))+G(z,T(z),T(z)), G(x,T(y),T(y))+G(y,T(x),T(x))+G(z,T(y),T(y)), G(x,T(z),T(z))+G(y,T(z),T(z))+G(z,T(x),T(x))\}\Big)\ \forall x,y,z \in X.$$

Suppose $\psi(t) < \frac{t}{4}\ \forall t > 0$. Then *T* has a unique fixed point.

3.14.Corollary:- (VATS et al.[17], Theorem 3.1) Let *(X,G)* be a complete G-metric space. Suppose $0 < k < \frac{1}{4}$ and *T* $:X \to X$ satisfies the following:

$$(3.14.1)\ G(T(x),T(y),T(z)) \le k\max\{G(x,T(x),T(x)),G(x,T(y),T(y)),G(x,T(z),T(z)), G(y,T(y),T(y)),G(y,T(x),T(x)),G(y,T(z),T(z)), G(z,T(z),T(z)),G(z,T(x),T(x)),G(z,T(y),T(y))\}\ \forall x,y,z \in X.$$

Then *T* has a unique fixed point.

Proof:- Take $\psi(t) = kt$ for $t \ge 0$.

Then $(3.14.1) \Rightarrow (3.13.1)$ and hence the result follows from Theorem 3.13

The following theorem can be proved on the lines of proof of Theorem 3.3

3.15 Theorem:- Let *(X,G)* be a complete G-metric space and $\psi \in \Psi$ is such that $\psi(t) < \frac{t}{2}$. Suppose *T* $:X \to X$ satisfies the following:

$$(3.15.1)\ G\left(T(x),T(y),T(z)\right)$$
$$\le \psi\Big(\max\big\{\max\{G(x,T(y),T(y))+G(y,T(x),T(x)),$$
$$G(y,T(z),T(z))+G(z,T(y),T(y)),$$
$$G(z,T(x),T(x))+G(x,T(z),T(z))\},$$
$$G(x,y,z),\{\max G(x,T(x),T(x)),G(y,T(y),T(y)),$$
$$G(z,T(z),T(z))\}\big\}\Big)$$
$$\forall x,y,z\in X.$$

Then T has a unique fixed point.

3.16 Corollary:- (Sushanta Kumar Mohanta[16], Theorem 3.4) Let *(X,G)* be a complete G-metric space, and $T:X\to X$ be such that

$$(3.16.1)\ G(T(x),T(y),T(z))$$
$$\le a\{G(x,T(y),T(y))+G(y,T(x),T(x))\}$$
$$+b\{G(y,T(z),T(z))+G(z,T(y),T(y))\}$$
$$+c\{G(z,T(x),T(x))+G(x,T(z),T(z))\}+d\ G(x,y,z)$$
$$+e\max\{G(x,T(x),T(x)),G(y,T(y),T(y)),G(z,T(z),T(z))\}$$

for all $x,y,z\in X$, where $a,b,c,d,e\ge 0$ with $2a+2b+2c+d+2e<1$.

Then T has a unique fixed point.

Proof:- Write

$$k=a+b+c+e+\frac{d}{2}\ \text{so that}\ k<\frac{1}{2}.$$

Take $\psi(t)=kt$ for $t\ge 0$.

Then $(3.16.1)\Rightarrow(3.15.1)$

and hence the result follows from Theorem 3.15

The proof of the following theorem is similar to that of Theorem 3.3

3.17 Theorem:- Let *(X,G)* be a complete G-metric space, and $T:X\to X$ be a mapping which satisfies the following condition

$$(3.17.1)\ G(T(x),T(y),T(z))$$
$$\le\psi\big(\max\{G(x,T(x),T(x)),G(y,T(y),T(y)),G(z,T(z),T(z)),$$
$$G(x,T(y),T(y)),G(y,T(z),T(z)),G(z,T(x),T(x)),$$
$$G(x,T(z),T(z)),G(y,T(x),T(x)),G(z,T(y),T(y)),$$
$$G(x,T(y),T(z)),G(y,T(z),T(x)),G(z,T(x),T(y)),$$
$$G(x,y,T(z)),G(y,z,T(x)),G(z,x,T(y)),G(x,y,z)\}\big)$$

for all

$x,y,z\in X$, where $\psi(t)<\frac{t}{3}$ for $t>0$.

Then T has a unique fixed point.

3.18 Corollary:- Let *(X,G)* be a complete G-metric space, and $T:X\to X$ be a mapping which satisfies the following condition

$$(3.18.1)\ G(T(x),T(y),T(z))$$
$$\le k\max\{G(x,T(x),T(x)),G(y,T(y),T(y)),G(z,T(z),T(z)),$$
$$G(x,T(y),T(y)),G(y,T(z),T(z)),G(z,T(x),T(x)),$$
$$G(x,T(z),T(z)),G(y,T(x),T(x)),G(z,T(y),T(y)),$$
$$G(x,T(y),T(z)),G(y,T(z),T(x)),G(z,T(x),T(y)),$$
$$G(x,y,T(z)),G(y,z,T(x)),G(z,x,T(y)),G(x,y,z)\}$$

for all $x,y,z\in X$.

Suppose $0\le k<\frac{1}{3}$. Then T has a unique fixed point

Proof:- Take $\psi(t)=kt$ for $t\ge 0$.

Then $(3.18.1)\Rightarrow(3.17.1)$

and hence the result follows from Theorem 3.17

The proof of the following theorem is similar to that of Theorem 3.3

3.19 Theorem:- Let *(X,G)* be a complete G-metric space, and $T:X\to X$ be a mapping which satisfies the following condition

$$(3.19.1)\ G(T(x),T(y),T(z)) \le \psi\big(\max\{G(x,T(y),T(y)),G(y,T(x),T(x)),G(z,T(z),T(z)), G(y,T(z),T(z)),G(z,T(y),T(y)),G(x,T(x),T(x)), G(z,T(x),T(x)),G(x,T(z),T(z)),G(y,T(y),T(y))\}\big)$$

for all $x, y, z \in X$, Suppose $\psi(t) < \frac{t}{4}$ for $t \ge 0$.

Then T has a unique fixed point.

3.20 Corollary:- (Sushanta Kumar Mohanta[16], Theorem 3.9) Let *(X,G)* be a complete G-metric space, and $T: X \to X$ be a mapping which satisfies the following.

Suppose

$$(3.20.1)\ G(T(x),T(y),T(z)) \le k\max\{G(x,T(y),T(y)),G(y,T(x),T(x)),G(z,T(z),T(z)), G(y,T(z),T(z)),G(z,T(y),T(y)),G(x,T(x),T(x)), G(z,T(x),T(x)),G(x,T(z),T(z)),G(y,T(y),T(y))\}$$

for all $x, y, z \in X$.

Suppose $0 \le k < \frac{1}{3}$. Then T has a unique fixed point.

Proof:- Take $\psi(t) = kt$ for $t \ge 0$.

Then $(3.20.1) \Rightarrow (3.19.1)$ and hence the result follows from Theorem 3.19

The proof of the following theorem is similar to that of Theorem 3.3

3.21 Theorem:- Let *(X,G)* be a complete G-metric space, and $T: X \to X$ be a mapping which satisfies the following condition

$$(3.21.1)\ G(T(x),T(y),T(z)) \le \psi\big(\max\{G(x,T(y),T(y)),G(y,T(x),T(x)),G(z,T(x),T(y)), G(y,T(z),T(z)),G(z,T(y),T(y)),G(x,T(y),T(z)), G(z,T(x),T(x)),G(x,T(z),T(z)),G(y,T(z),T(x))\}\big)$$

for all $x, y, z \in X$. Suppose $\psi(t) < \frac{t}{4}$ for $t \ge 0$.

Then T has a unique fixed point.

3.22 Corollary:- (Sushanta Kumar Mohanta[16], Theorem 3.11)Let *(X,G)* be a complete G-metric space, and $T: X \to X$ suppose

$$(3.22.1)\ G(T(x),T(y),T(z)) \le k\max\{G(x,T(y),T(y)),G(y,T(x),T(x)),G(z,T(x),T(y)), G(y,T(z),T(z)),G(z,T(y),T(y)),G(x,T(y),T(z)), G(z,T(x),T(x)),G(x,T(z),T(z)),G(y,T(z),T(x))\}$$

for all $x, y, z \in X$.

Suppose $0 \le k < \frac{1}{4}$. Then T has a unique fixed point.

Proof:- Take $\psi(t) = kt$ for $t \ge 0$.

Then $(3.22.1) \Rightarrow (3.21.1)$ and hence the result follows from Theorem 3.21

The following theorem can be proved as in Theorem 3.3

3.23 Theorem:- Let *(X,G)* be a complete G-metric space, and $T: X \to X$ be a mapping which satisfies the following condition

$$(3.23.1)\ G(T(x),T(y),T(z))$$
$$\le \psi\Big(\max\{G(x,T(y),T(y)),G(y,T(x),T(x)),G(x,y,T(z)),$$
$$G(y,T(z),T(z)),G(z,T(y),T(y)),G(y,z,T(x)),$$
$$G(z,T(x),T(x)),G(x,T(z),T(z)),G(z,x,T(y))\}\Big)$$

for all $x, y, z \in X$.

Suppose $\psi(t) < \frac{t}{5}$ for $t \ge 0$. Then T has a unique fixed point.

3.24 Corollary:- (Sushanta Kumar Mohanta [16], Theorem 3.12) Let *(X,G)* be a complete G-metric space, and $T: X \to X$ be a mapping which satisfies the following condition

$$(3.24.1)\ G(T(x),T(y),T(z))$$
$$\le k\max\{G(x,T(y),T(y)),G(y,T(x),T(x)),G(x,y,T(z)),$$
$$G(y,T(z),T(z)),G(z,T(y),T(y)),G(y,z,T(x)),$$
$$G(z,T(x),T(x)),G(x,T(z),T(z)),G(z,x,T(y))\}$$

for all $x, y, z \in X$.

Suppose $0 \le k < \frac{1}{5}$. Then T has a unique fixed point.

Proof:- Take $\psi(t) = kt$ for $t \ge 0$.

Then $(3.24.1) \Rightarrow (3.23.1)$

and hence the result follows from Theorem 3.23

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